Math 245B Lecture 11 Notes

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1 Introduction to Norms and Normed Vector Spaces

1.1 Normed vector spaces

First, here is our notation. We will denote $K = \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{X}$ to be a vector space over K. We will denote $Kx = \{\lambda x : \lambda \in K\}$ and 0 as the origin in K or $\mathcal{X}$. If $\mathcal{M}, \mathcal{N}$ are vect spaces in $\mathcal{X}$, then we denote $\mathcal{M} + \mathcal{N} = \{x + y : x \in \mathcal{M}, y \in \mathcal{N}\}$.

Definition 1.1. A **seminorm** on $\mathcal{X}$ is a function $\|\cdot\| : \mathcal{X} \to [0, \infty)$ such that

1. $\|x + y| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$
2. $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in \mathcal{X}$ (homogeneous of order 1).

A **norm** is a seminorm such that $\|x\| = 0 \implies x = 0$. A pair $(\mathcal{X}, \|\cdot\|)$ is a **normed vector space**.

The second property of seminorms implies that $\|0\| = 0$.

Definition 1.2. The **norm metric** on $(\mathcal{X}, \|\cdot\|)$ is $\rho(x, y) = \|x - y\|$. This generates the **norm topology**.

This is the kind of definition

Example 1.1. $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean norm are normed vector spaces.

Example 1.2. The space $BC(X, K)$ with $\|f\|_u := \sup_{x \in X} |f(x)|$.

Example 1.3. The space $\ell_K^\infty = \{(x_n)_{n=1}^\infty \in K^{\mathbb{N}} : \sup_n |x_n| < \infty\}$ is a normed vector space with the norm $\|x\|_\infty = \sup_n |x_n|$. This is actually $BC(\mathbb{N}, K)$.

Example 1.4. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $L_K^1(\mu)$, the set of measurable functions $f : X \to K$ such that $\|f\|_1 = \int |f| \, d\mu < \infty$, is not a normed vector space. In fact, $\|\cdot\|_1$ is a seminorm, so to get a normed vector space, we need to look at equivalence classes of functions that agree μ-a.e.

Example 1.5. The space $\ell^1_K = \{(x_n)_n \in K^{\mathbb{N}} : \|x_1\| = \sum_n |x_n| < \infty\}$ is a normed vector space.

Example 1.6. $\ell^2_K = \{(x_n)_n \in K^{\mathbb{N}} : \|x\|_2^2 = \sum_n |x_n|^2 < \infty\}$ is a normed vector space. In fact, if we replace 2 by p for $1 \leq p < \infty$, we also get a normed vector space.

1.2 Completeness and convergence

Definition 1.3. A **Banach space** over K is a normed vector space over K which is complete in the norm metric.

All the above examples are Banach spaces.

Example 1.7. Here is an incomplete Banach space.[1] Let $Y = \{x \in \ell^1_K : \exists n_0 \in \mathbb{N} \text{ s.t. } x_n = 0 \, \forall n \geq n_0\}$.

Definition 1.4. A series $\sum_{n=1}^\infty x_n$ in $(\mathcal{X}, \|\cdot\|)$ is **convergent** if there exists some $x \in \mathcal{X}$ such that $\|x - \sum_{n=1}^N x_n\| \to 0$ as $N \to \infty$. It is **absolutely convergent** if $\sum_{n=1}^\infty \|x_n\| < \infty$.

Proposition 1.1. *A normed space $(\mathcal{X}, \|\cdot\|)$ is complete if and only if every absolutely convergent sequence is convergent.*

Proof. ($\Longrightarrow$): Assume $\mathcal{X}$ is complete. Let $S_N = \sum_{n=1}^n x_n$. Then for $M > N$,

$$\|S_N - S_m\| = \left\| \sum_{n=N+1}^M x_n \right\| \leq \sum_{n=N+1}^M \|x_n\| \xrightarrow{N,M\to\infty} 0.$$

Then S_N is Cauchy, which means it has a limit.

($\Longleftarrow$): Suppose $(x_n)_n$ is Cauchy. Then $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. Pick $n_1 < n_2 < \cdots$ such taht $\|x_n - x_m\| < 2^{i-1}$ for all $n, m \geq n_j$. Define $y_1 = x_{n_1}$ and $y_j = x_{n_j} - x_{n_{j-1}}$ for $j \geq 2$. Note that $\sum_{j=1}^k y_j = x_{n_k}$. Also,

$$\sum_{j=1}^k \|y_j\| = \|x_{n_1}\| + \sum_{j=2}^k \|x_{n_j} - x_{n_{j-1}}\| \leq \|x_{n_1}\| + \sum_{j=2}^\infty 2^{-(j-1)} < \infty.$$

So there exists some $x = \lim_{k\to\infty} \sum_{j=1}^k y_j = \lim_k x_{n_k}$. Then $x_n \to x$. $\square$

Remark 1.1. In this proof, we used a very useful technique: pass to a subsequence to upgrade the convergence to a much faster convergence.

Proposition 1.2. *$L^1_K(\mu)$ is complete.*

[1] If you ever wonder whether a property is using the completeness of a Banach space, try seeing if it still holds in this space.

Proof. Assume $(f_j)_j \in L^1_{\mathbb{R}}(\mu)$ such that $\sum_j \int |f_j|\, d\mu < \infty$. Let $g_N = \sum_{j=1}^N |f_j|$ be non-negative and increasing in N. By the monotone convergence theorem, there exists some g such that $g = \lim_N g_N$, $g \geq 0$, and $\int g = \lim \int g_N < \infty$. Now if $F_N = \sum_{j=1}^N f_j$, then $|F_N| \leq g$. Moreover, $\sum_{j=N}^M |f_j| \leq g - g_N \to 0$ whenever $g < \infty$ (which holds a.e.). So $F(x) := \lim_N F_N(x)$ exists for a.e. x.By the dominated convergence theorem, we conclude that $\int F_N\, d\mu \to \int F\, d\mu$. Similarly, $|F_N - F| \leq 2g$ be the triangle inequality, and $|F_N - F| \to 0$ pointwise. So by the dominated convergence theorem again, $\int |F_N - F| \to \int 0 = 0$.

The case $K = \mathbb{C}$ is similar. □

1.3 Norms over finite dimensional vector spaces

Definition 1.5. Two norms $\|\cdot\|$ and $\|\cdot\|'$ are **equivalent** if there exists some $C \in (0, \infty)$ such that $(1/C)\|x\| \leq \|x\|' \leq C\|x\|$ for all $x \in \mathcal{X}$.

Theorem 1.1. *If* $\dim(\mathcal{X}) < \infty$*, all norms are equivalent.*

Proof. We will treat the $K = \mathbb{R}$ case; the $K = \mathbb{C}$ case is similar. It is enough to show this when $\mathcal{X} = \mathbb{R}^n$. Let $|\cdot|$ be the Euclidean norm and $\|\cdot\|$ be another norm. We will show that $|\cdot\|$ and $\|\cdot\|$ are equivalent.

Let $e_1, \dots, e_n$ be the standard basis. Then

$$\|x\| = \left\| \sum_{i=1}^n x_i e_i \right\| \leq \sum_{i=1}^n |x_i| \|e_i\| \leq \left(\sum_i |x_i|^2 \right)^{1/2} \left(\sum_i \|e_i\|^2 \right)^{1/2}$$

by Cauchy-Schwarz. In fact, this shows that $\|\cdot\|$ is continuous.

To finish, it is enough to show that $\inf\{\|x\| : |x| = 1\} > 0$. But this infimum is achieved at some x such that $|x| = 1$. We must still have $\|x\| > 0$ at this x. □

The proof also showed us the following.

Corollary 1.1. $\|\cdot\|$ *is continuous for the usual topology on* $\mathbb{R}^n$*.*